

On the calculation of Fisher information for quantum parameter estimation based on the stochastic master equation

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ABSTRACT

The Fisher information can be used to indicate the precision of parameter estimation by the quantum Cramér-Rao inequality. This paper presents an efficient numerical algorithm for the calculation of Fisher information based on quantum weak measurement. According to the quantum stochastic master equation, the Fisher information is expressed in the form of log-likelihood functions. Three main methods are employed in this algorithm: (i) we use the numerical differentiation approach to calculate the derivative of the log-likelihood function; (ii) we randomly generate a series of parameters of interest by the Metropolis Hastings (MH) algorithm; and (iii) the values of expectation can be approximated by the Markov chain Monte Carlo (MCMC) integration. Finally, as an example to testify the feasibility of the proposed algorithm, we consider the dissipation rates of the open quantum system as unknown parameters that need to be estimated. We show that the Fisher information can reach a precision close to the Heisenberg limit in the weak coupling condition. This again demonstrates the effectiveness of the new algorithm.

Introduction

In reality, not all parameters in the open quantum system^{1,2} can be obtained directly. Moreover, there are always some inevitable estimation errors in those who need to be estimated. How to reduce the errors has become a key problem in recent years. The parameter estimation theory³⁻⁵ tells us that any parameter estimation has an estimated precision and it's hard to surpass the limit precision by the traditional methods. In the classical case, the maximum precision is called the standard quantum limit⁶ or shot noise limit, $1/\sqrt{N}$, where N represents the number of experiments or that of general particle experiments. However, Caves^{7,8} showed that with the help of squeezed state technique, quantum mechanical systems can achieve greater sensitivity over the standard quantum limit. Theoretically the ultimate precision limit is the Heisenberg limit⁹ $1/N$. The classic Fisher information was originally used to describe the information of unknown parameter contained in a random variable. It's well-known that the variance of any unbiased estimation is at least as high as the inverse of the Fisher information¹⁰⁻¹³. This is also known as the Cramér-Rao inequality, $\text{Var}\hat{\theta} \geq 1/I(\theta)$. The Fisher information provides a better way to calculate the estimation precision.

Since quantum projective measurements will in general disturb the system they are measuring, the Fisher information may undergoes a large deviation. Improving the accuracy of quantum parameter estimation based on quantum weak measurement¹⁴⁻¹⁶ caused a wide range of interests. Smith and co-authors proposed a protocol to achieve fast, accurate and non-destructive quantum state estimation based on continuous weak measurement in the presence of a controlled dynamical evolution¹⁷. Xu and co-authors used a weak measurement scheme to realize high precision quantum phase estimation¹⁸. Gammelmark and Mølmer derived a likelihood function to estimate the unknown parameters with the help of quantum measurement and quantum stochastic master equation¹⁹. They investigated the statistical properties of the output state, which will provide the ultimate limits in estimation precision. Although much progress has been made in quantum parameter estimation based on continue weak measurement, how to effectively calculate the Fisher information based on the quantum stochastic master equation is still with remarkable difficulty. To figure out this problem, one needs to represent the Fisher information in computable forms and take effective measures to prior-estimate the parameter of interest. Recently, Genoni proposed a method to calculate the Fisher information for linear Gaussian quantum system, whose evolution depends only on the evolution of first and second moments of the quantum states²⁰.

In this paper, we propose an efficient numerical algorithm to calculate the Fisher information based on the quantum stochastic master equation. Three main methods are employed in this algorithm: (i) we use the numerical differentiation approach to calculate the derivative of the log-likelihood function; (ii) we randomly generate a series of parameters of interest by the Metropolis Hastings (MH) algorithm²¹; and (iii) the values of expectation can be approximated by the Markov chain Monte

Carlo (MCMC) integration²². Moreover, we reduce the complexity of calculation by showing that any quantum stochastic master equation for un-normalized states corresponds to the evolutions of some normalized states. The result of this work opens an efficient way to obtain the ultimate precision of parameter estimation for the open quantum system.

Results

Stochastic master equation based on quantum weak measurement

We consider the quantum parameter estimation of the open quantum system based on quantum weak measurement. The unknown parameters may exist in the system Hamiltonian, the dissipation rates, and coupling or measurement strength. Here, the measurement process is assumed to be a Markov process. During the measurement and estimation processes, the quantum stochastic master equation method has always been involved. For brevity, the stochastic master equation^{23–25} based on quantum weak measurement for an un-normalized state $\tilde{\rho}_t$ is given by

$$d\tilde{\rho}_t = -i[H, \tilde{\rho}_t]dt + \left(L\tilde{\rho}_t L^\dagger - \frac{1}{2}(L^\dagger L\tilde{\rho}_t + \tilde{\rho}_t L^\dagger L) \right) dt + \sqrt{\eta}(L\tilde{\rho}_t + \tilde{\rho}_t L^\dagger) dY_t, \quad (1)$$

where H is the Hamiltonian of the quantum system, η is the measurement strength with the weak measurement constraint $\eta \ll 1$, and dY_t is the infinitesimal increment which represents the measurement output. Based on the relationship between an un-normalized quantum state $\tilde{\rho}_t$ and a normalised state ρ_t , $\rho_t = \tilde{\rho}_t / \text{Tr}(\tilde{\rho}_t)$, we have

$$dY_t = \sqrt{\eta} \text{Tr}(\rho_t L + L^\dagger \rho_t) dt + dW_t, \quad (2)$$

where dW_t is the Winner increment with zero mean and variance dt . It describes the quantum fluctuations of the continuous output signal. For convince, we introduce a map $M(\rho) = L\rho + \rho L^\dagger$, and a likelihood function $\mathcal{L}_t = \text{Tr}(\tilde{\rho}_t)$. Below, we derive a log-likelihood function which is closely related to Fisher information and stands for the precision of parameter estimation.

According to Eq. (1), the derivative of the likelihood function \mathcal{L}_t with respect to time t can be written as

$$d\mathcal{L}_t = \text{Tr}(d\tilde{\rho}_t) = \sqrt{\eta} \text{Tr}(M(\tilde{\rho}_t)) dY_t = \sqrt{\eta} \text{Tr}(M(\rho_t)) \mathcal{L}_t dY_t. \quad (3)$$

Thus, we can obtain the normalized quantum stochastic master equation by means of the multi-dimensional Itô formula.

$$d\rho_t = -i[H, \rho_t]dt + \left(L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) \right) dt + \sqrt{\eta}(M(\rho_t) - \rho_t \text{Tr}(M(\rho_t))) dW_t. \quad (4)$$

Quantum Fisher information

Suppose θ is an unknown parameter of the open quantum system that needs to be estimated. As mentioned above, the Fisher information can be used to indicate the precision of parameter estimation by the quantum Cramér-Rao inequality^{26–28}, i.e.,

$$\langle (\delta\hat{\theta})^2 \rangle \geq \frac{1}{NI(\theta)}, \quad (5)$$

where $I(\theta)$ is the Fisher information and N is the number of measurements. Obviously, if $I(\theta)$ approaches N , it means that the estimation precision is closed to the Heisenberg limit. Below, we use l_t to denote the log-likelihood function^{29,30}, and we have

$$dl_t = d\ln \mathcal{L}_t = \frac{d\mathcal{L}_t}{\mathcal{L}_t} = \sqrt{\eta} \text{Tr}(M(\rho_t)) dY_t. \quad (6)$$

Therefore, the Fisher information can be rewritten as

$$I(\theta) = E \left[\left(\frac{\partial \ln \mathcal{L}_t}{\partial \theta} \right)^2 \right] = E \left[\left(\frac{\partial l_t}{\partial \theta} \right)^2 \right]. \quad (7)$$

Substituting Eq. (6) into Eq. (7), we can obtain the analytic form of the quantum Fisher information.

Calculation of the quantum Fisher information

From the Fisher information Eq. (7), it is easy to find that θ is not an independent variable of the likelihood function. In other words, it is not an explicit expression and that makes the calculation with remarkable difficulty. In order to efficiently calculate the quantum Fisher information, we propose a numerical algorithm with the help of MH algorithm and MCMC integration.

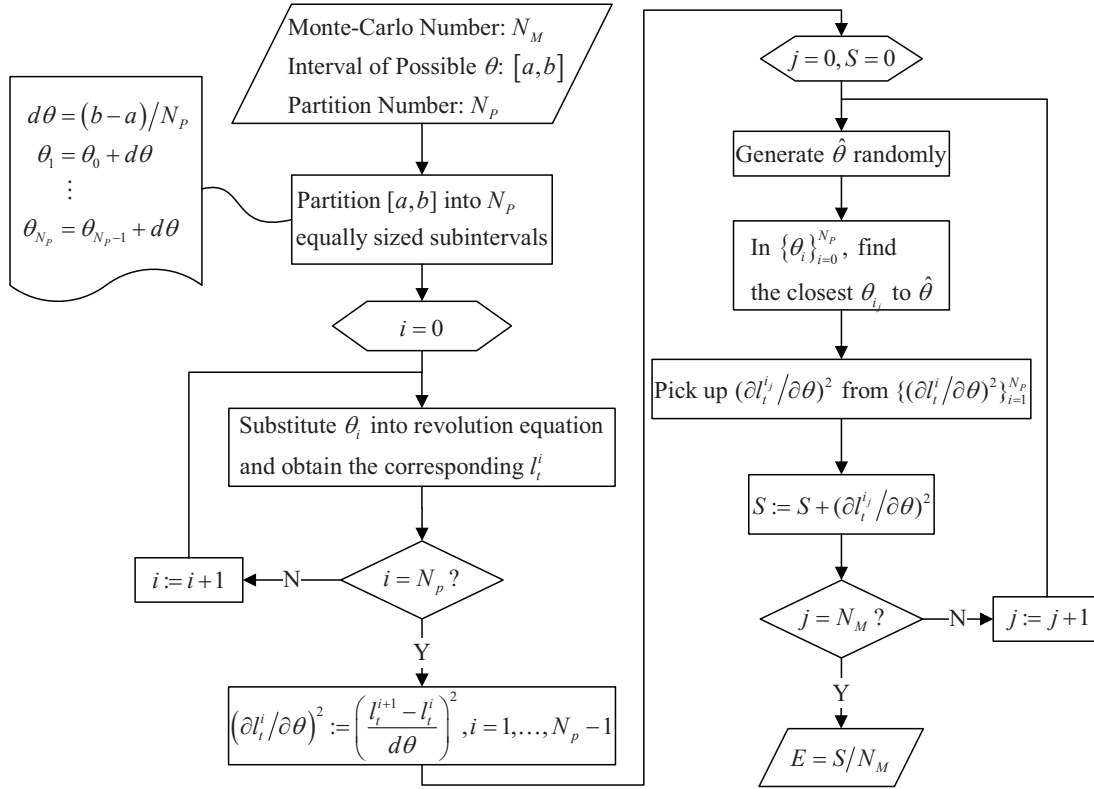


Figure 1. The procedure of calculating the quantum Fisher information based on quantum weak measurement.

In the beginning, we make the continuous attributes of the unknown parameter that needs to be estimated discrete, $\theta_1, \dots, \theta_{N_p}$, which satisfy

$$\theta_{i+1} = \theta_i + d\theta, \quad i = 1, 2, \dots, N_p. \quad (8)$$

Here the interval $d\theta$ is a constant. For each θ_i , we have the corresponding log-likelihood functions l_t by means of the quantum stochastic master equation. Here, the functions $l_t^1, l_t^2, \dots, l_t^{N_p}$ are the functions of time t with $t \in [0, T]$. Owing to the basic definition of the derivative, the differential of l_t with respect to θ is approximately given by

$$\frac{dl_t}{d\theta} = \frac{l_t^{i+1} - l_t^i}{\theta_{i+1} - \theta_i} = \frac{l_t^{i+1} - l_t^i}{d\theta}, \quad i = 1, 2, \dots, N_p - 1. \quad (9)$$

In order to efficiently calculate the Fisher information, we randomly generate a cluster of $\hat{\theta}$ by the MH algorithm^{21,22} (one can refer to the Method for details), whose prior probability distribution is assumed to satisfy a certain distribution. Thus, we have

$$\hat{\theta} = \{\hat{\theta}_j | j = 1, 2, \dots, N_M\}, \quad (10)$$

where N_M is the number of measurements. However, what we need to notice is that the number of using candidate points N_A that used to generate random samples is larger than the number of measurement, i.e., $N_M \leq N_A$. In the set Eq. (10), the fluctuation of the pre-estimated parameter values is rather small. This process makes the following calculation as close to real as possible. For simplicity, one may anticipate the initial value of the sequence generating $\hat{\theta}$ to be a constant value. It is easy to find this value by comparing $\hat{\theta}$ with θ and choose the closest θ_{i_j} to each $\hat{\theta}_j$. Accordingly, $\partial l_t / \partial \theta$ could be determined based on the generated $\hat{\theta}$ that are picked out from log-likelihood function.

On the basis of Eq. (7), calculating the Fisher information means to acquire the expected value $E[(\partial l_t / \partial \theta)^2]$ from the sample $\partial l_t^1 / \partial \theta, \partial l_t^2 / \partial \theta, \dots, \partial l_t^{N_M} / \partial \theta$. By the Markov chain Monte Carlo integration^{21,22}, see Method for details, the Fisher information can be approximated as

$$E[(\partial l_t / \partial \theta)^2] = E\left[\left(\frac{dl_t}{d\theta}\right)^2\right] \approx \frac{1}{N_M} \sum_{j=1}^{N_M} \left(\frac{\partial l_t^j}{\partial \theta}\right)^2. \quad (11)$$

As a conclusion, the procedure of calculating the quantum Fisher information is shown in Figure 1.

An example: estimation of the dissipation rate in an open quantum system

We consider a single two-level atom in a coherently driven cavity and damped by spontaneous emission of photons^{1,31}. The evolution of the system satisfies the above stochastic master equation Eq. (1). The Hamiltonian of the system and the Lindblad operator are

$$H = \frac{\Omega}{2}\sigma^x + \frac{\Delta}{2}\sigma^z, \quad C = \sqrt{\gamma}L = \sqrt{\gamma}i\sigma^-, \quad (12)$$

where Ω, Δ, γ are the Rabi-frequency, the detuning and the dissipation rate, respectively. Here, the unknown parameter that need to be estimated is the dissipation rate.

According to Eq. (1), the evolution of the un-normalized state $\tilde{\rho}_t$ can be described as

$$d\tilde{\rho}_t = -i[H, \tilde{\rho}_t]dt + \gamma \left(L\tilde{\rho}_t L^\dagger - \frac{1}{2}(L^\dagger L\tilde{\rho}_t + \tilde{\rho}_t L^\dagger L) \right) dt + \sqrt{\eta\gamma}(L\tilde{\rho}_t + \tilde{\rho}_t L^\dagger)dY_t, \quad (13)$$

As the same as Eq. (4), we have the evolution of the normalized system state

$$d\rho_t = -i[H, \rho_t]dt + \gamma \left(L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) \right) dt + \sqrt{\eta\gamma}(M(\rho_t) - \rho_t \text{Tr}(M(\rho_t)))dW_t. \quad (14)$$

To calculate the quantum Fisher information, one just needs to compute l_t . Based on the evolution with the normalized state ρ_t , it is easy to figure out this problem.

Finally, the numerical calculation of the quantum Fisher information can be achieved by programming the flow diagram, Fig. 1.

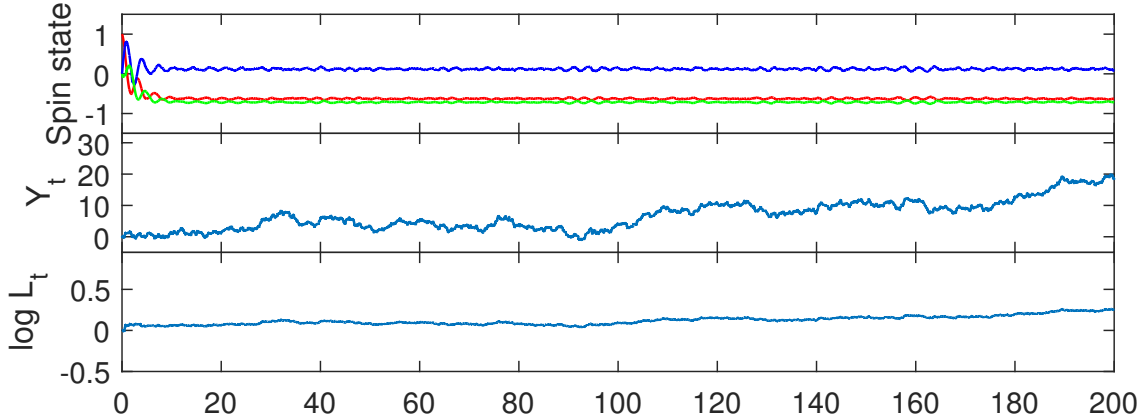


Figure 2. The top figure shows the evolution of the three components of the Bloch vector during one measurement period. Here, the curves are $x = \text{Tr}(\sigma^x \rho_t)$ (red), $y = \text{Tr}(\sigma^y \rho_t)$ (blue), and $z = \text{Tr}(\sigma^z \rho_t)$ (green). The middle one plots the measurement output Y_t . The bottom shows the log-likelihood function $\log L_t$.

Numerical results

We denote the system state ρ_t as

$$\rho_t = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \quad (15)$$

and the initial state is $x(0) = 1, y(0) = z(0) = 0$. The parameters in the stochastic master equation are $\Omega = 1.3, \Delta = 1.43$ ¹⁹ and $\eta = 0.01$. The initial value of the sequence generating the unknown parameter γ is assumed to be 0.55, whose stationary distribution and proposal distribution are assumed to satisfy two certain distributions $N(0, 1)$ and $N(0, (dt)^2)$, respectively. Fig. 2 shows the evolution of the normalized state during one measurement period. In Fig. 2, we also plot the output Y_t and the log-likelihood function $\log L_t$.

Based on the proposed algorithm, in Fig. (1), we simulate the whole process for the calculation of the Fisher information in Fig. 3, where both generation and selection of the parameters are included. Here, the number of measurement is 500, i.e.,

$N_M = 500$. Fig. 3(a) shows the choice of γ and $\hat{\gamma}$. In Fig. 3(b), we plot several trajectories of $(dl_t/d\gamma)^2$. Fig. 3(c) demonstrates the evolution of the Fisher information on average. Here, the maximum Fisher information is 447.3154, which is close to the Heisenberg limit 500. This simulation shows that we can quickly calculate the quantum Fisher information. Finally, this example shows the feasibility and effectiveness of the proposed algorithm.

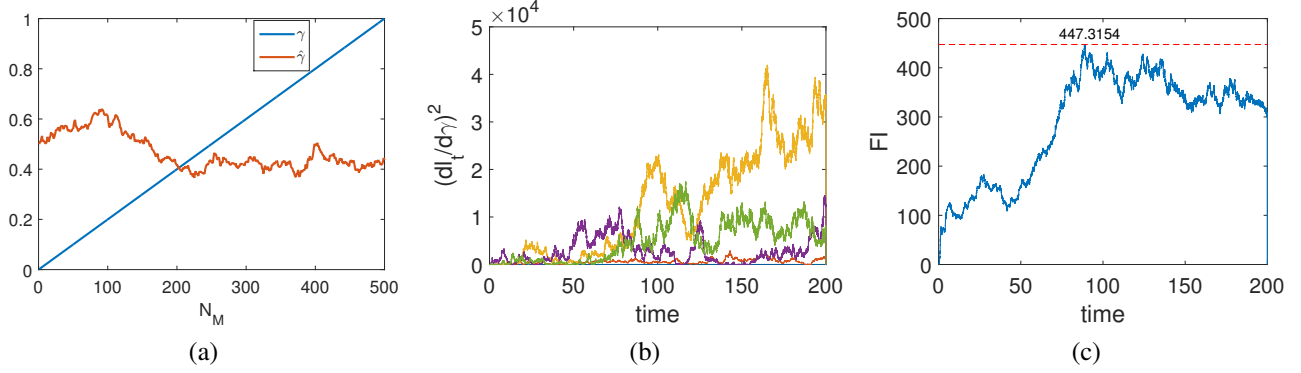


Figure 3. Fig. (a) shows the choice of γ and $\hat{\gamma}$. In Fig. (b), we plot several trajectories of $(dl_t/d\gamma)^2$. Fig. (c) shows the evolution of the Fisher information on average. Here, the maximum Fisher information is 447.3154, which is close to the Heisenberg limit 500.

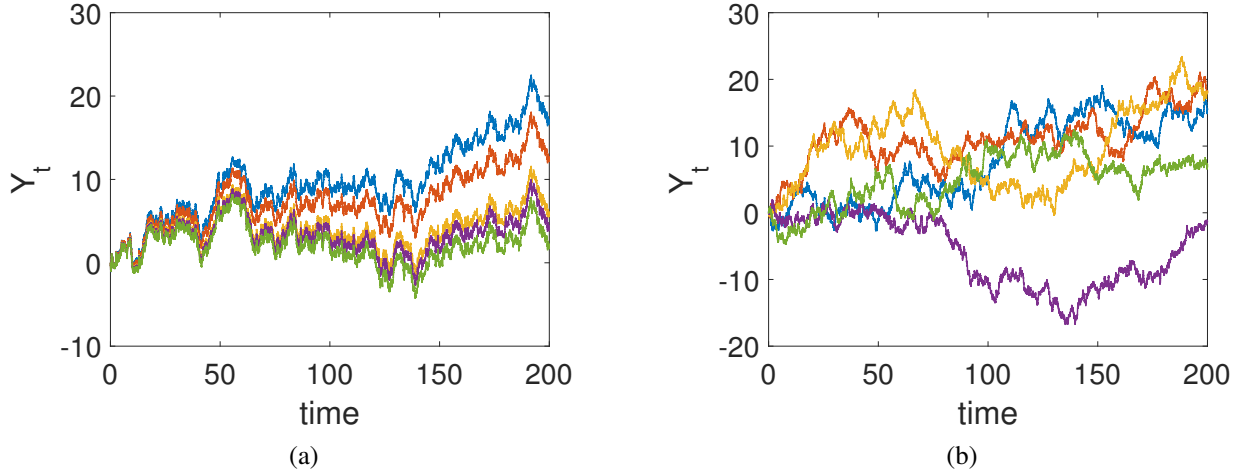


Figure 4. The evolution of the output function Y_t for various measurement strength η , 1.0(blue), 0.5(red), 0.1(orange), 0.05(purple), and 0.01(green). Fig. (a) plots the evolutions with the same random quantity. In Fig. (b), the random quantities would vary with the measurement strength η .

Discussion

We have already demonstrated the effectiveness of the new algorithm in calculating the Fisher information with quantum weak measurement. A question arises on what is the threshold of the measurement strength η . As we mentioned in the beginning, strong measurement would totally destroy the quantum system. Thus, it is of particular interest to find the threshold, below which one can both accurately calculate the quantum Fisher information and protect the system from being destroyed by the quantum measurement. In this work, we simply set η as 0.01 and the simulation results are in agreement with the theoretical calculations. In Fig. 4 and Fig. 5, we plot the evolution of the output function Y_t and the log-likelihood function l_t for various measurement strength η . Obviously, the stochastic quantity has a great influence on the output function Y_t , which decreases with η under the same Winner process. However, the log-likelihood function l_t just depends on η . When η is less than a certain value, the smaller the Wiener process fluctuates the smaller the log-likelihood function. Searching for the exact value of this threshold still remains to be further studied.

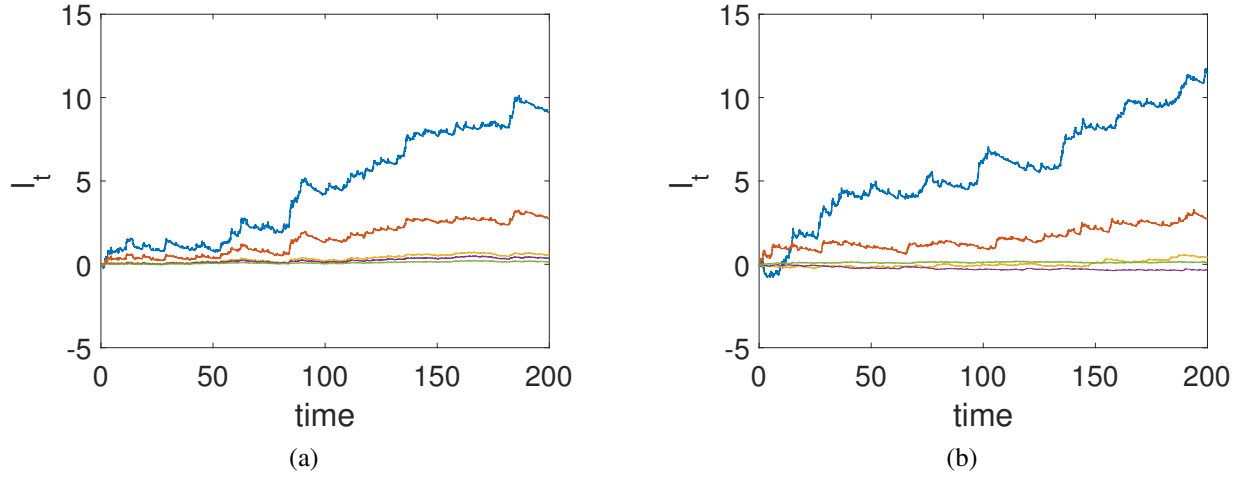


Figure 5. The evolution of the log-likelihood function l_t for various measurement strength η , 1.0(blue), 0.5(red), 0.1(orange), 0.05(purple), and 0.01(green). Fig. (a) plots the evolutions with the same random quantity. In Fig. (b), the random quantities would vary with the measurement strength η .

Methods

Metropolis Hastings algorithm²¹

In Markov chains, suppose we generate a sequence of random variables X_1, X_2, \dots, X_n with Markov property, namely the probability of moving to the next state depends only on the present state and not on the previous state:

$$\Pr\{X_{n+1} = x | X_1 = x_1, \dots, X_n = x_n\} = \Pr\{X_{n+1} = x | X_n = x_n\}.$$

Then, for a given state X_t , the next state X_{t+1} does not depend further on the hist of the chain X_1, X_2, \dots, X_{t-1} , but comes from a distribution which only on the current state of the chain X_t . For any time instant t , if the next state is the first sample reference point Y obeying distribution $q(\bullet | X_t)$ which is called the transition kernel of the chain, then obviously it depends on the current state X_t . In generally, $q(\bullet | X_t)$ may be a multidimensional normal distribution with mean X , so the candidate point Y is accepted with probability $\alpha(X_t, Y)$ where

$$\alpha(X, Y) = \min\left(1, \frac{\pi(Y)q(X|Y)}{\pi(X)q(Y|X)}\right). \quad (16)$$

Here, $\pi(\hat{Y})$ stands a function only depends on \hat{Y} . If the candidate point is accepted, the next state becomes $X_{t+1} = Y$. If the candidate point is rejected, it means that the chain does not move, the next state will be $X_{t+1} = X$. We illustrate this sampling process with a simple example, see Fig. 6. Here, the initial value is $X(1) = -10$. Fig. 6(a) represents the stationary distribution $N(0, 0.1)$. In Fig. 6(b), we plot 500 iterations from Metropolis Hastings algorithm with the stationary distribution $N(0, 1)$ and proposal distribution $N(0, 0.1)$. Obviously, sampling data selecting from the latter part would be better.

Makov Chain Monte Carlo integration²²

In Markov chain, we can use the Monte Carlo integration to evaluate $E[f(X)]$ by drawing samples $\{X_1, X_2, \dots, X_n\}$ from Metropolis Hastings algorithm. Here

$$E[f(X)] \approx \frac{1}{n} \sum_{i=1}^n f(X_i), \quad (17)$$

means that the population mean of $f(X)$ is approximated by the sample mean. When the sample X_i are independent, law of large numbers ensures that the approximation can be made as accurate as desired by increasing the sample. Note that here n is not the total amount of samples by Metropolis Hastings algorithm but the length of drawing samples.

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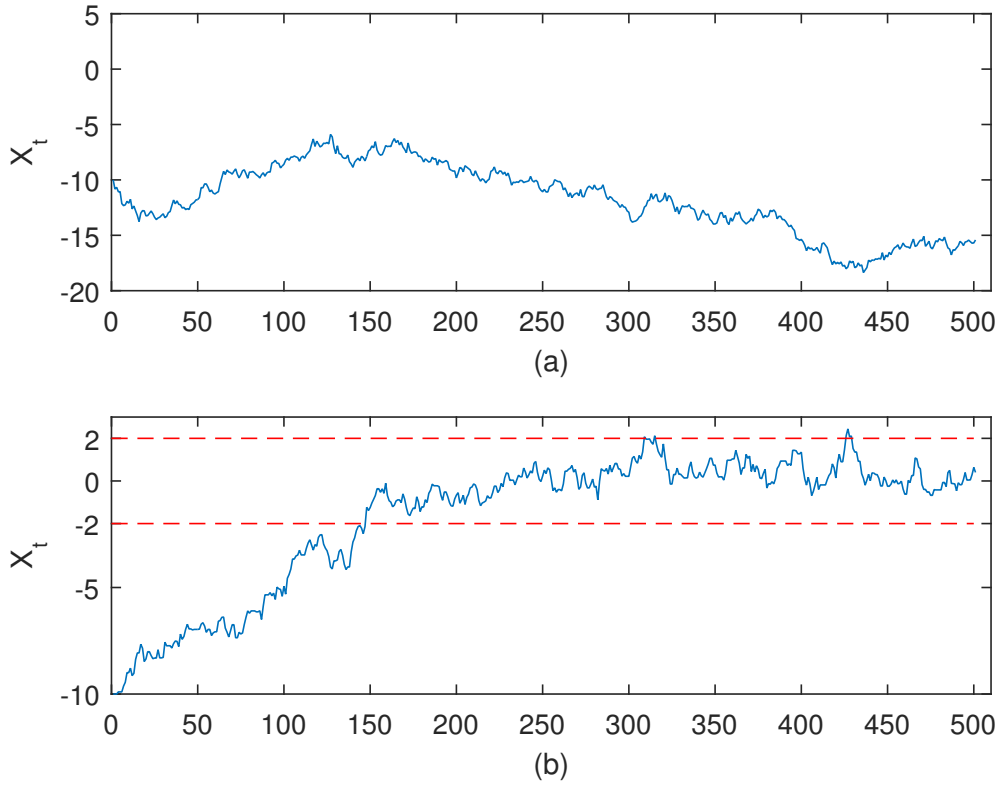


Figure 6. We illustrate the sampling process of the Metropolis Hastings algorithm. Here, the initial value is $X(1) = -10$. Fig. (a) represents the stationary distribution $N(0, 0.1)$. In Fig. (b), we plot 500 iterations from Metropolis Hastings algorithm with the stationary distribution $N(0, 1)$ and proposal distribution $N(0, 0.1)$. Obviously, sampling data selecting from the latter part would be better.

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Author contributions statement

B.G. and W.C. contributed to design the ideas, perform the calculations, analyse the results and write the manuscript.

Additional information

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